SECOND TERM EXAMINATION, 2011–2012

MATHEMATICS

Time Allowed : 3 hrs.  

General Instructions : 
- All questions are compulsory.
- Section A : Q. 1. to Q. 10 are of 1 mark each.
- Section B : Q. 11 to Q. 22 are of 4 marks each.
- Section C : Q. 23 to Q. 29 are of 6 marks each.

SECTION–A

1. If \( f(x) \) is an invertible function, find the inverse of \( f(x) = \frac{5x - 3}{7} \).

2. If \( \sin^{-1} x - \cos^{-1} x = \frac{\pi}{6} \), then solve for \( x \).

3. If \( A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix} \), write \( A^{-1} \) in terms of \( A \).

4. For what value of \( x \), the matrix \( \begin{bmatrix} 5 - x & x + 1 \\ 2 & 4 \end{bmatrix} \) is singular?

5. For what values of ‘\( a \)’ the vectors \( 2\hat{i} - 3\hat{j} + 4\hat{k} \) and \( a\hat{i} + 6\hat{j} - 8\hat{k} \) are collinear?

6. Find the value of \( \lambda \) such that the line \( \frac{x-2}{9} = \frac{y-1}{\lambda} = \frac{z-3}{-6} \) is perpendicular to the plane \( 3x - y - 2z = 7 \).

7. Find the distance of the point \( (a, b, c) \) from \( x \)-axis.

8. If \( f(x) = \sin x^0 \), find \( \frac{dy}{dx} \).

9. Evaluate \( \int \frac{1 - \tan x}{1 + \tan x} \, dx \).

10. If \( x = 2 \cos \theta - \cos 2\theta \) 
    \( y = 2 \sin \theta - \sin 2\theta \)
    find \( \frac{dy}{dx} \) at \( \theta = \frac{\pi}{2} \).

SECTION–B

11. A binary operation \( * \) on the set \( \{0, 1, 2, 3, 4, 5\} \) is defined as:
    \[ a * b = \begin{cases} 
    a + b, & \text{if } a + b < 6 \\
    a + b - 6, & \text{if } a + b \geq 6
    \end{cases} \]

P.T.O.
Show that zero is that identity for this operation each element ‘a’ of the set is invertible with 6 – a, being the inverse of ‘a’.

12. Prove that:
\[
\tan^{-1}\left[\frac{\sqrt{1 + x} - \sqrt{1 - x}}{\sqrt{1 + x} + \sqrt{1 - x}}\right] = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x, \text{ when } -\frac{1}{\sqrt{2}} \leq x \leq 1.
\]

13. Prove using properties of determinants
\[
\begin{vmatrix}
 y+k & y & y \\
 y & y+k & y \\
 y & y & y+k \\
\end{vmatrix} = k^2 (3y+k)
\]

Or

If \( A \) and \( B \) are square matrices of the same order such that \( AB = BA \), then prove by induction that \( (AB)^n = B^n A \). Further, prove that \( (AB)^n = A^n B^n \) \( \forall n \in \mathbb{N} \).

14. Show that \( f(x) = |x - 3|, \ x \in \mathbb{R} \) is continuous, but not differentiable at \( x = 3 \).

15. Show that the function \( f(x) = \cot^{-1} (\sin x + \cos x) \) is a strictly decreasing function on the interval \( \left(0, \frac{\pi}{4}\right) \).

16. Evaluate:
\[
\int \frac{4x+5}{\sqrt{2x^2 + x - 3}} \, dx.
\]

17. Evaluate:
\[
\int_{-1}^{1/2} |x \cos (\pi x)| \, dx.
\]

18. Solve:
\[
x \frac{dy}{dx} - ay = x + 1.
\]

19. Show that the differential equation \( (x-y) \frac{dy}{dx} = x + 2y \) is homogeneous and solve it.

20. If the sum of two unit vectors is a unit vector, show that the magnitude of their difference is \( \sqrt{3} \).

21. If \( \vec{a}, \vec{b}, \text{ and } \vec{c} \) are three unit vectors such that \( \vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c} = 0 \) and angle between \( \vec{b} \) and \( \vec{c} \) is \( \frac{\pi}{6} \), prove that \( \vec{a} = \pm 2 (\vec{b} \times \vec{c}) \).

22. A biased die is twice as likely to show an even number as an odd number. The die is rolled three times. If occurrence of an even number is considered a success, then write the probability distribution of the number of successes. Also find the mean number of success.
23. Use product
\[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 2 & -3 \\
3 & -2 & 4
\end{bmatrix}
\begin{bmatrix}
-2 & 0 & 1 \\
9 & 2 & -3 \\
6 & 1 & -2
\end{bmatrix}
\]
to solve the system of equations
\[
\begin{align*}
x - y + 2z &= 1 \\
2y - 3z &= 1 \\
3x - 2y + 4z &= 2
\end{align*}
\]
Or
Using elementary transformations, find the inverse of the matrix.
\[
\begin{bmatrix}
2 & 0 & -1 \\
5 & 1 & 0 \\
0 & 1 & 3
\end{bmatrix}
\]

24. From a pack of 52 cards, a card is lost. From the remaining 51 cards, two cards are drawn at random (without replacement) and are found to be both diamonds. What is the probability that the lost card was a card of heart?

25. There are two factories located one at place \( P \) and the other at place \( Q \). From these locations, a certain commodity is to be delivered to each of the three depots situated at \( A \), \( B \) and \( C \). The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at \( P \) and \( Q \) are respectively 8 and 6 units. The cost of transportation per unit is given below:

<table>
<thead>
<tr>
<th>From/to</th>
<th>( A )</th>
<th>( B )</th>
<th>( C )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( P )</td>
<td>160</td>
<td>100</td>
<td>150</td>
</tr>
<tr>
<td>( Q )</td>
<td>100</td>
<td>120</td>
<td>100</td>
</tr>
</tbody>
</table>

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

26. Find the equation of the plane passing through the point \( (1, 1, 1) \) and containing the line
\[
\vec{r} = (-3\hat{i} + \hat{j} + 5\hat{k}) + \lambda (\hat{i} - \hat{j} - 5\hat{k})
\]
Also, show that the plane contains the lines
\[
\vec{r} = (-\hat{i} + 2\hat{j} + 5\hat{k}) + \mu (\hat{i} - 2\hat{j} - 5\hat{k})
\]

27. Make a rough sketch of the region given below and find its area using integration:
\[
\{(x, y) : 0 \leq y \leq x^2 + 3; 0 \leq y \leq 2x; 0 \leq x \leq 3\}
\]
Or
\[
(3)
\]
Find the area of smaller region bounded by the ellipse \( \frac{x^2}{16} + \frac{y^2}{9} = 1 \) and the straight line \( \frac{x}{4} + \frac{y}{3} = 1 \).

28. Show that the semi-vertical angle of right circular cone of given surface area and maximum volume is \( \sin^{-1}\left(\frac{1}{3}\right) \).

29. Evaluate:

\[
\int_{0}^{\sqrt{2}} \frac{\sin^2 x \, dx}{\sin x + \cos x}.
\]

Or

\[
\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}} \, dx, \ x \in [0, 1].
\]